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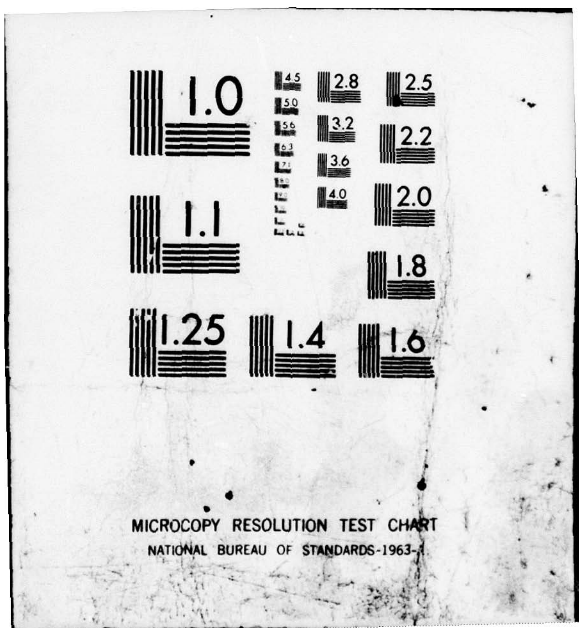
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NEAR-TIP PLASTIC DEFORMATIONS IN

DYNAMIC FRACTURE PROBLEMS

J. D. Achenbach, P. Burgers and V. Dunayevsky

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ABSTRACT

Under rapid loading conditions and/or for a rapidly propagating crack, the mass density of a material affects the fields of stress and deformation. For such dynamic fracture problems plastic deformations in the immediate vicinity of a crack tip are investigated in this paper. Both stationary and propagating crack tips are considered. For a stationary crack tip, deformation theory is employed for the first phase of the loading when the fields are increasing monotonically with time. The general character of the near-tip fields is analyzed both with respect to its variation with time and with polar angle. The non-linear near-tip fields are related to the linearly elastic far-field by means of a path-independent integral. In the second part of the paper we consider rapidly propagating cracks. We discuss the near-tip fields for various models of material behavior. In particularly we briefly review some earlier work by Achenbach and Kanninen for a rapidly propagating Mode-III crack, in a material which displays strain hardening. In the last part of the paper we consider the fields near a rapidly propagating crack-tip in an elastic-perfectly plastic material for the case that inertial terms are of importance. The system of governing equations in the plastic region is presented and shown to be hyperbolic in nature. As a first approximation the steady state case with respect to the moving crack-tip is considered and an asymptotic analysis of the near-tip field is carried out.

INTRODUCTION

There are two broad classes of fracture mechanics problems that may have to be treated as dynamic problems. These are concerned with: (1) cracked bodies subjected to rapidly varying loads, (2) bodies containing rapidly propagating cracks. In both cases the crack tip is in an environment of rapidly varying fields of stress and deformation.

Impact and vibration problems fall into the first class of dynamic problems. In the analysis of such problems it is often found that the dynamic stresses near flaws are higher than the stresses computed from the corresponding problem of static equilibrium. The dynamic stress "overshoot" can be especially pronounced for cracks. In view of the dynamic amplification, it is conceivable that there are cases for which fracture at a crack tip does not occur under a gradually applied system of loads, but where a crack does indeed propagate when the same system of loads is rapidly applied,

and gives rise to waves, which strike the crack tip.

The second class of problems is equally important, since there are several kinds of large engineering structures in which rapid crack growth is a definite possibility. When a crack propagates rapidly, dynamic effects affect the stress fields near the crack tip, and hence the conditions for further unstable crack propagation or for crack arrest.

In recent years there have been a number of comprehensive review articles in the general area of elastodynamic fracture mechanics, see Refs.[1]-[5]. At present, dynamic fracture mechanics solutions are, however, largely confined to conditions where linear elastic fracture mechanics (LEFM) is valid. The elastic-plastic treatments required to give accurate results have not yet been developed in a completely acceptable manner, even under static conditions. Current progress in this area, and a starting point for the development of a dynamic plastic propagating crack tip analysis have recently been discussed in Ref.[6].

In this paper further investigations are reported on plastic deformations in the immediate vicinity of a crack tip for both stationary and moving crack tips.

The analysis for a stationary crack tip is based on deformation theory. For the corresponding class of static problems deformation theory was applied by Hutchinson [7] and Rice and Rosengren [8]. In Section 3 some of the results of [7] and [8] are extended to dynamic near-tip fields. Deformation theory is not valid for unloading, and consequently the results generally apply only in an initial time interval when the fields of stress and deformation increase monotonically. Also, since the nonlinear near-tip fields are related to the linear far-field by means of a path-independent integral, the plastic deformation must be confined to a small zone near the crack tip.

In Sections 4 and 5 plastic deformations near a rapidly propagating crack tip are considered, for the special case of Mode III deformations. The analysis takes the inertia terms into account, but it is assumed that the fields are steady-state, i.e., they are constant with respect to an observer who moves with the crack-tip. The corresponding quasi-static problem has been considered by Chitaley and McClintock [9] for an elastic perfectly-plastic material, and by Amazigo and Hutchinson [10] for the case of J_2-flow theory with a bilinear effective stress-strain curve.

Section 4 is primarily concerned with a discussion of the influence of various constitutive behaviors on the near-tip fields. In this section we also briefly review some earlier results of Achenbach and Kanninen [6] for J_2-flow theory. In particularly we investigate the nature of the governing equations as the crack-tip velocity increases. For the case of strain hardening the governing equations are elliptic when the crack-tip velocity is not too large. In that case the usual separation of variables asymptotic analysis yielding singular stress and strain fields can be carried out, and singularities of the type $r^p(-1 < p < 0)$ are obtained. As the velocity increases (or the strain-hardening curve becomes flatter) the nature of the equations becomes hyperbolic. It appears very difficult to trace the transition of the near-tip fields as the velocity increases.

LINEAR ELASTODYNAMIC STRESS INTENSITY FACTORS

It is well known that for linearly elastic materials, the stress fields in the immediate vicinity of a stationary or moving crack tip can be expressed in the general form

$$\sigma_{ij} = r^{-\frac{1}{2}} k(t,v)\, \Sigma_{ij}(\theta,v)\ , \qquad (1)$$

where r,θ are polar coordinates centered at the crack tip, as shown in Fig.1. v is the instantaneous velocity of the crack tip and σ_{ij} is the Cauchy stress tensor. The functions $\Sigma_{ij}(\theta,v)$ are universal functions in that they are independent of the overall geometry of the body and particular loading systems. It is only through a single parameter, the stress intensity factor $k(t,v)$, that the overall geometry and loading of the body influence the near

tip stress fields. The form given in Eqn.(1) is valid for propagating cracks and stationary cracks ($v = 0$) under both dynamic and quasi-static loading. It is also applicable to three dimensional problems of a plane crack with a smoothly curved edge, if the (r,θ)-plane is taken normal to the crack edge. The radius of curvature of the crack edge then enters of course in the magnitudes of the stress intensity factors. The components of $\Sigma_{ij}(\theta,v)$ show a significant dependence on v. It is of note that for the Mode-I case the maximum of $\Sigma_{\theta\theta}(\theta,v)$ moves out of the plane of the crack when v increases beyond a certain value.

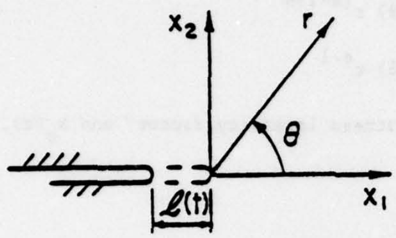

Fig. 1 Coordinate system for a propagating crack tip

DYNAMIC FIELDS FOR STATIONARY CRACK TIPS USING DEFORMATION THEORY

In this section, an incompressible elastic power-law hardening material is considered. The analysis is carried out for the case of plane strain and the usual small strain-displacement relations are used.

The dynamic Mode-I stress intensity factor, $k_I(t)$, for a stationary crack tip in a linearly elastic material can be expressed as

$$k_I(t) = f(t)\, K_I \tag{2}$$

where K_I is the corresponding quasi-static stress intensity factor. It will now be shown that the dynamic stress intensity factor for the non-linear material can be expressed in terms of a function of f(t), which is defined by Eq.(2), and the corresponding non-linear static fields.

Near Tip Fields

Following Ref.[8], an incompressible material is considered, with a relation between deviatoric stresses, s_{ij} and strains, ϵ_{ij} of the form

$$s_{ij} = (2\tau/\gamma)\, \epsilon_{ij} \tag{3}$$

where $\tau = (s_{ij}s_{ij}/2)^{\frac{1}{2}}$ and $\gamma = (2\epsilon_{ij}\epsilon_{ij})^{\frac{1}{2}}$. The hardening is assumed to be governed by

$$\tau = \mu\,\gamma = \left(\frac{\tau_o}{\gamma_o}\right)\gamma \quad \text{for } \gamma \le \gamma_o$$
$$\tau = \tau_o\left(\frac{\gamma}{\gamma_o}\right)^N \quad \text{for } \gamma \ge \gamma_o\,, \tag{4}$$

where τ_o, γ_o are, respectively, the yield stress and strain in shear, and N is the hardening exponent satisfying $0 < N \le 1$. Since the plastic strains are incompressible and only the dominant singularity is considered, the assumption of material incompressibility should not be a poor approximation for moderate strain hardening. The system of governing equations is completed with the equations of motion

$$\sigma_{ij,j} = \rho\ddot{u}_i\,. \tag{5}$$

To obtain the dominating singularity at the crack tip, the near tip field is expanded in terms of powers of r and functions of t and θ. The displacements are expressed as

$$u_i \to k_\epsilon(t)\, \tilde{u}_i(r,\theta) = k_\epsilon(t)\, U_i(\theta)\, r^q \text{ as } r \to 0 \tag{6}$$

where q > 0. After use of Eqns.(3) and (4), corresponding expressions for the stresses and strains, are obtained as

$$\sigma_{ij} \to k_\sigma(t)\, \tilde{\sigma}_{ij}(r,\theta) = k_\sigma(t)\, \Sigma_{ij}(\theta)\, r^{(q-1)N} , \tag{7}$$

$$\epsilon_{ij} \to k_\epsilon(t)\, \tilde{\epsilon}_{ij}(r,\theta) = k_\epsilon(t)\, E_{ij}(\theta)\, r^{q-1} , \tag{8}$$

as $r \to 0$, where $k_\sigma(t)$ is the "dynamic stress intensity factor" and $k_\sigma(t)$, $k_\epsilon(t)$ are related by

$$k_\sigma(t) = [k_\epsilon(t)]^N. \tag{9}$$

Substituting Eqns.(6) and (9) into the equations of motion (5) shows that the highest order singularities on the left hand-side of Eqn.(5) are of order $r^{(q-1)N-1}$, while those on the right-hand side are of order r^q. This implies, as in the linear elastic case, that for a stationary crack the inertia terms do not affect the variation with respect to r and θ of the near-tip fields to the highest order of singularity. The results for q, $u_i(\theta)$, $\Sigma_{ij}(\theta)$ and $E_{ij}(\theta)$ are the same for static and dynamic problems, i.e. the H.R.R. singular fields

$$\sigma_{ij} = O(r^{-N/(1+N)}) \text{ and } \epsilon_{ij} = O(r^{-1/(1+N)}),$$

dominate in the vicinity of the crack tip.

Specific results for the functions $u_i(\theta)$, $\Sigma_{ij}(\theta)$ and $E_{ij}(\theta)$ are given in Refs.[7] and [8].

A Path-Independent Integral

For the case of linear materials, Nilsson [11] has presented a path-independent integral, of the form

$$I_\ell = \int_C \left[\left(\frac{1}{2}\,\bar{\sigma}_{ij}\bar{\epsilon}_{ij} + \frac{1}{2}\,\rho\, p^2 \bar{u}_i \bar{u}_i\right) n_1 - \bar{T}_i \frac{\partial \bar{u}_i}{\partial x_1}\right] ds \tag{10}$$

in terms of Laplace transforms of the field variables. The contour C defines a path enclosing the crack tip, n_i are the components of the outer normal to C, a superscript bar indicates the Laplace transform and p is the Laplace transform variable.

For material behavior described by nonlinear deformation theory, and in a region where Eqns.(6),(7) and (8) are valid, a functional $W(\tilde{\epsilon}_{ij})$ can be defined by

$$W(\tilde{\epsilon}_{ij}) = \int_0^{\tilde{\epsilon}_{ij}} \tilde{\sigma}_{ij}(\tilde{\epsilon}^*_{ij})\, d\tilde{\epsilon}^*_{ij} . \tag{11}$$

The following integral

$$I_{n\ell} = \int_C \left[(k_\sigma\, k_\epsilon\, W(\tilde{\epsilon}_{ij}) + \frac{1}{2}\,\rho\, p^2\, \bar{u}_i\, \bar{u}_i)\, n_1 - \bar{T}_i\, \partial \bar{u}_i / \partial x_1\right] ds \tag{12}$$

is then path independent as $r \to 0$. To prove this we consider a closed contour C^*, with enclosed area A^* and we let I^* represent $I_{n\ell}$ along C^*.

By the divergence theorem,

$$I^* = \int_{A^*} \left[\bar{k}_\sigma \, \bar{k}_\varepsilon \, \frac{\partial W(\tilde{\varepsilon}_{ij})}{\partial x_1} + \frac{1}{2} \rho \, p^2 \frac{\partial}{\partial x_1} (\bar{u}_i \bar{u}_i) - \frac{\partial}{\partial x_j} \left(\bar{\sigma}_{ij} \frac{\partial \bar{u}_i}{\partial x_1}\right)\right] dA \tag{13}$$

where $\bar{T}_i = \bar{\sigma}_{ij} \, n_j$ has been used. Now

$$\bar{k}_\sigma \, \bar{k}_\varepsilon \, \frac{\partial W(\tilde{\varepsilon}_{ij})}{\partial x_1} = \bar{\sigma}_{ij} \, \bar{k}_\varepsilon \, \frac{\partial \tilde{\varepsilon}_{ij}}{\partial x_1} = \bar{\sigma}_{ij} \frac{\partial}{\partial x_1} \left(\frac{\partial \bar{u}_i}{\partial x_j}\right) \tag{14}$$

where $\tilde{\sigma}_{ij} = \partial W(\tilde{\varepsilon}_{km})/\partial \tilde{\varepsilon}_{ij}$ and the symmetry of $\tilde{\sigma}_{ij}$ has been used. Substitution of Eqn.(14) into Eqn.(13) and use of the Laplace transform of the equation of motion, shows $I^* = 0$. Realizing that the fields used apply only as $r \to 0$, this implies that $I_{n\ell}$ is path independent in the vicinity of the crack tip.

The assumption of small scale yielding is now introduced, i.e. the region of nonlinear deformation is assumed small as compared to the region in which the linear elastic singularity dominates, which in turn is small with respect to any characteristic dimension of the body [12]. Since $I_{n\ell}$ is valid in both the linear and nonlinear regions close to the crack tip, it can be used to connect the nonlinear elastodynamic near-tip field to the surrounding linear elastodynamic field.

For the linear elastic case, I_ℓ can be expressed in terms of the Laplace transform of the dynamic stress intensity factor [11]. For plane strain:

$$I_\ell = \left[\bar{k}_1(p)\right]^2 (1 - \nu^2)/E \, , \tag{15}$$

and

$$I_{n\ell} = \bar{k}_\sigma(p) \, \bar{k}_\varepsilon(p) \, \tilde{J} + p^2 \left[\bar{k}_\varepsilon(p)\right]^2 \tilde{L} \tag{16}$$

where

$$\tilde{J} = \int_C \left[W(\tilde{\varepsilon}_{ij}) \, n_1 - \tilde{T}_i \frac{\partial \tilde{u}_i}{\partial x_1}\right] ds, \tag{17}$$

$$\tilde{L} = \frac{1}{2} \rho \int_C \tilde{u}_i \, \tilde{u}_i \, dx_2 \tag{18}$$

and $\bar{T}_i = \bar{k}_\sigma(p) \, \tilde{T}_i$. In the limit as C shrinks on to the crack tip, $\tilde{L}$ vanishes.

Intensity Factors

By the reasoning given above, we have $I_\ell = I_{n\ell}$ which gives

$$\bar{k}_\sigma(p) \, \bar{k}_\varepsilon(p) \, \tilde{J} = \left[\bar{k}_1(p)\right]^2 (1 - \nu^2) / E \, . \tag{19}$$

Anologously to Eqn.(2) we introduce

$$\left\{\bar{k}_\sigma(p); \, \bar{k}_\varepsilon(p)\right\} = \left\{\bar{f}_\sigma(p) \, k_\sigma \, ; \, \bar{f}_\varepsilon(p) \, k_\varepsilon\right\} \tag{20}$$

where K_σ , K_ε are the intensity factors for the corresponding static problem. Using the quasi-static result that $K_\sigma \, K_\varepsilon \, \tilde{J} = K_1^2 \, (1 - \nu^2)/ E$ and Eqns(2) and (18), Eqn.(19) is reduced to

$$\bar{f}_\sigma(p) \, \bar{f}_\varepsilon(p) = \left[\bar{f}(p)\right]^2 \tag{21}$$

which, together with (9), relates the time dependence of the dynamic stress intensity factors using deformation theory to the time dependence of the

corresponding intensity factors using linear elasticity.

To understand the implications of Eqn.(21), let $f(t) = t^{\alpha}$ ($\alpha > 0$), i.e. $\bar{f}(p) = \Gamma(\alpha+1)/p^{(\alpha+1)}$ where Γ is the gamma function. The corresponding forms for f_ϵ, f_σ are $f_\epsilon(t) = C\,t^{\delta}$, $f_\sigma(t) = C^N\,t^{N\delta}$. Substituting the Laplace transform of $\bar{f}_\epsilon$, $\bar{f}_\sigma$ into Eqn.(21) gives

$$C^{N+1}\,\frac{\Gamma(\delta N+1)}{p^{\delta N+1}}\,\frac{\Gamma(\delta+1)}{p^{\delta+1}} = \left[\frac{\Gamma(\alpha+1)}{p^{\alpha+1}}\right]^2 \tag{22}$$

from which we find $\delta = 2\alpha/(N+1)$. Since $0 < N \leq 1$, it follows that $\delta > \alpha$ which implies that $f_\epsilon(t) \sim t^{\delta}$ increases slower than $f(t)$, but $f_\sigma(t) \sim t^{N\delta}$ increases faster than for the linearly elastic case.

INFLUENCE OF CONSTITUTIVE BEHAVIOR ON NEAR-TIP FIELDS

The functional dependence on r and θ of the fields of stress and deformation near a rapidly propagating crack tip can generally be established by asymptotic considerations. For a wide class of constitutive behaviors an asymptotic analysis can be based on "separation of variables" near-tip solutions, in which the dependence on r is a-priori assumed as a power of r. The analysis proceeds by collecting the most singular terms in the governing equations, and the boundary conditions, and subsequently solving the resulting linear or nonlinear eigenvalue problem. When this method is applicable it is found that the stresses and strains are singular.

In an asymptotic analysis only the field in the highly strained material in the near-tip region is considered. Hence, it is the stress-strain behavior at very large strains which enters an asymptotic analysis. The speed of crack propagation is limited by the characteristic speed of the material immediately ahead of the crack tip. This speed is related to the slope of the stress-strain curve. Thus, only constitutive models with a finite slope of the stress-strain curve at large strains are suitable for the type of asymptotic analysis described above.

In this section the influence of the constitutive model on the near-tip field is examined for rapidly propagating cracks. Only the case of Mode III crack propagation (antiplane strain) has so far been investigated *in some* detail.

The asymptotic analysis is carried out in a moving coordinate system, which is fixed with respect to the crack tip. If the speed of crack propagation is $v = d\ell(t)dt$, where $\ell(t)$ is crack length and v is an arbitrary function of time subject to the conditions that v and dv/dt be continuous, the material time derivatives are transformed to the moving coordinate systems by the relations

$$(\dot{\ }) = \frac{\partial}{\partial t} - v(t)\,\frac{\partial}{\partial x_1}$$

and

$$(\ddot{\ }) = \frac{\partial^2}{\partial t^2} - \dot{v}(t)\,\frac{\partial}{\partial t} - 2\,v(t)\,\frac{\partial^2}{\partial t\,\partial x_1} + v^2(t)\,\frac{\partial^2}{\partial x_1^2} \tag{23}$$

In the usual separation of variable type asymptotic analysis considered here, the terms coming from $\partial^2 u_i/\partial x_1^2$ will be more singular than any other terms in $\ddot{u}_i$, so that the results will be the same for the "steady state" and transient cases as far as the r and θ dependence is concerned. By "steady state", it is meant that, as seen by a crack-tip observer, the stress and strain fields are constant.

Motion in antiplane strain is defined by a displacement in the x_3-direction only, see Fig. 1. The notation is simplified by writing.

$w = u_{x_3}(x_1, x_2)$, $u_{x_1} = u_{x_2} = 0$,

$$\tau_i = \sigma_{3i} , \qquad i = 1, 2$$

$$\gamma_i = 2\epsilon_{3i} = \partial w/\partial x_i \; . \qquad i = 1, 2 \tag{24}$$

The equations of motion and compatability reduce to

$$\tau_{i,i} = \rho \ddot{w} \;\; , \gamma_{1,2} = \gamma_{2,1} \;\; , \tag{25}$$

where $\ddot{w}$ is found from Eqn.(23).

The system of governing equations must be completed by the constitutive relations. In the following we examine the near-tip fields for the various different constitutive behaviors shown in Fig. 2.

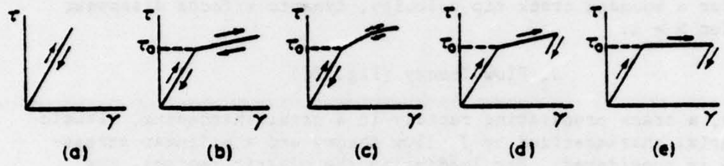

Fig. 2 Models of material behavior

Linear Elasticity (Fig. 2a)

The constitutive law is $\tau_i = \mu \gamma_i$ and the near-tip field is of the form given in section 2. That is the stresses and strains have the familiar square root singularity with respect to r, of LEFM. The upper limit for crack propagation under remotely applied loads is the elastic shear wave speed $(\mu/\rho)^{\frac{1}{2}}$.

Bilinear elasticity (Fig. 2b)

Defining an effective shear stress, τ, and shear strain, γ, as $\tau = (\tau_i \; \tau_i)^{\frac{1}{2}}$, $\gamma = (\gamma_i \gamma_i)^{\frac{1}{2}}$, the constitutive law is given by

$$\mu \gamma_i = \tau_i \;\; , \qquad \tau \le \tau_o$$

$$\mu \gamma_i = \tau_i + (\mu/\mu_t - 1)(1 - \tau_o/\tau) \; \tau_i \;\; , \; \tau > \tau_o \tag{26}$$

where μ is the linear elastic shear modulus and $\mu_t = d\tau/d\gamma$ for $\tau > \tau_o$

The singularities for stress and strain are similar to the ones for classical linear elasticity, but with μ_t replacing μ as the relevant elastic constant. This reduction in the slope of the stress-strain curve has important consequences for the significance of dynamic effects in rapid crack propagation. Although the crack speed v may be small compared to the linear elastic shear-wave speed, $(\mu/\rho)^{\frac{1}{2}}$, it may be a significant fraction of the characteristic wave speed at high strains, $(\mu_t/\rho)^{\frac{1}{2}}$, and this is the important comparison in deciding the importance of dynamic effects. If the fracture process is essentially brittle, $(\mu_t/\rho)^{\frac{1}{2}}$ is an approximate upper limit on the crack propagation velocity.

Power-law Strain Hardening (Fig. 2c)

The Mode III version for deformation theory of plasticity is

$$\tau_i = \mu \gamma_i , \qquad \tau \le \tau_o$$

$$\tau_i = \tau_o (\gamma/\gamma_o)^N \gamma_i , \quad \tau > \tau_o \tag{27}$$

where τ_o, γ_o are the yield stress and strain respectively and $N > 0$.

The loading path in stress space of a material point close to the tip of a rapidly propagating crack is not proportional since a zone of unloading exists. Deformation theory of plasticity will therefore be a poor model in this region. Even if Eqn.(27) is assumed to be valid for a non-linear elastic material, it is not possible to obtain an asymptotic expansion in powers of r. For $0 < N < 1$, $\mu_t(\tau) = d\tau/d\gamma$ vanishes as $\gamma \to \infty$, and the characteristic wave speed becomes zero. Therefore, any speed of crack propagation is supersonic and the above asymptotic expansion is not valid. For $N > 1$, $d\tau/d\gamma$ becomes unbounded as $\gamma \to \infty$ and so does the characteristic wave speed. This means that for a bounded crack tip velocity, dynamic effects disappear altogether for $N > 1$.

J_2 Flow Theory (Fig. 2d)

Finally, a crack propagating rapidly in a strain-hardening, elastic-plastic material characterized by J_2 flow theory and a bilinear stress-strain curve is considered. For loading in the plastic regions, the incremental stress-strain relations are

$$\mu_t \dot{\gamma}_i = \alpha \dot{\tau}_i + (1 - \alpha)(\dot{\tau}/\tau) \tau_i , \quad \tau > \tau_o , \quad \dot{\tau} > 0 , \tag{28}$$

while for elastic unloading and elastic loading we have

$$\mu_t \dot{\gamma}_i = \alpha \dot{\tau}_i , \quad \dot{\tau} \le 0 , \text{ or } \tau \le \tau_o , \tag{29}$$

where $\alpha = \mu_t/\mu$.

If the steady-state situation is considered, the above equations together with the equations of motion and compatability are not always elliptic. When the governing equations are elliptic, an asymptotic expansion in powers of r will be valid. If the material time derivatives are replaced by $- v\, \partial/\partial x_1$ and the compatability equation in Eqn.(25) is used to eliminate γ_2 from the constitutive equation (28), the governing equations in the plastic loading region can be written as the following system of first order equations,

$$\begin{bmatrix} -v^2\rho & 1 & 0 \\ 1 & -[1 + \tau_1^2 (1/\alpha - 1)]/(\mu\tau^2) & -\tau_1\tau_2(1/\alpha - 1)/(\mu\tau^2) \\ 0 & -\tau_1\tau_2(1/\alpha - 1)/(\mu\tau^2) & -[1 + \tau_2^2(1/\alpha - 1)]/(\mu\tau^2) \end{bmatrix} \underline{w}_{,1} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \underline{w}_{,2} = \underline{0} \tag{30}$$

where

$$\underline{w} = \begin{Bmatrix} \gamma_1 \\ \tau_1 \\ \tau_2 \end{Bmatrix}$$

Using the form $A^1 \underline{w}_{,1} + A^2 \underline{w}_{,2} = \underline{0}$ to define the matrices A^1, A^2, the characteristic directions [13] (where characteristics are defined as lines across which discontinuities in $\underline{w}$ can occur) are given by $\Lambda = dx_2/dx_1$. These directions are found by solving the characteristic equation

$$|A^2 - \Lambda A^1| = 0 \quad . \tag{31}$$

The system of equations is elliptic if the roots Λ are complex.

Equation (31) can be written in the form $\Lambda f(\Lambda) = 0$, where $f(\Lambda)$ is quadratic in Λ, with roots $\Lambda^{\pm}$ which will be complex if the discriminant of $f(\Lambda)$ is negative. This condition can be reduced to

$$[v^2/(\mu/\rho)][1 + \tau_1^2 (1/\alpha - 1)/\tau^2] - 1 < 0 \tag{32}$$

If $\tau_1 = \tau$, a lower bound of

$$v < (\mu_t/\rho)^{\frac{1}{2}} \tag{33}$$

is found for the above system of equations to be elliptic [14], which is the limit given earlier on the basis of intuitive reasoning.

For crack tip velocities less than this limit, the separation of variables approach to an asymptotic expansion will be valid. However, for higher velocities, the type of the governing equations depends on the magnitude of τ_1.

The case of $v < (\mu_t/\rho)^{\frac{1}{2}}$ will be considered below, and in the next section the elastic perfectly-plastic case will be considered. Following Achenbach and Kanninen [6], an asymptotically valid solution for $\dot{w}$ of the general form

$$\dot{w} = C\, v\, \dot{W}(\theta)\, r^s \tag{34}$$

is sought. Here C is an amplitude factor, left undetermined by the analysis, and $\dot{W}(\theta)$ and s have to be found. The problem is set up as a generalized eigenvalue problem with s as the eigenvalue.

Using Eqn.(24), the strain rates corresponding to Eqn.(33) are

$$\dot{\gamma}_i = C\, v \frac{\partial}{\partial x_1}\left[\dot{W}(\theta)\, r^s\right] \quad . \quad i = 1,2 \tag{35}$$

Since only the most singular terms in r will be retained in the asymptotic analysis, in Eqn.(23) quantities of the form $\partial f/\partial t$, where f is any field variable, may be neglected when compared with $-v(t)\, \partial f/\partial x_1$, so that $(\dot{\ }) \simeq -v\, \partial/\partial x_1$. In the asymptotic analysis v is considered to be constant.

The stresses corresponding to Eqn.(24) are

$$(\tau_i, \tau) = \mu\, C\, (T_i, T)\, r^s \; , \tag{37}$$

which implies

$$T = (T_1^2 + T_2^2)^{\frac{1}{2}} \; . \tag{38}$$

The stress rates are defined by

$$(\dot{\tau}_i, \dot{\tau}) = -\mu\, C\, v \frac{\partial}{\partial x_1}\left[\left(T_i, T\right) r^s\right] \simeq \mu\, C\, v\, (\dot{T}_i, \dot{T})\, r^{s-1} \tag{39}$$

Substituting Eqns.(34) and (37) into the equation of motion, Eqn.(35), and retaining only the most singular terms, yields

$$s\, T_1 \cos\theta - T_1' \sin\theta + s\, T_2 \sin\theta + T_2' \cos\theta = \beta^2(-s\dot{W} \cos\theta + \dot{W}' \sin\theta) \tag{40}$$

where superscript ' indicates $\partial/\partial\theta$ and $\beta^2 = v^2/(\mu/\rho)$.

Substitution of Eqns.(33) and (37-39) into Eqn.(28) yields

$$\alpha(s\,\dot{W}\cos\theta - \dot{W}'\sin\theta) = \alpha\,\dot{T}_1 + (1-\alpha)\,T^{-1}\,T_1\,\dot{T} \tag{41}$$

$$\alpha(s\,\dot{W}\sin\theta + \dot{W}'\cos\theta) = \alpha\,\dot{T}_2 + (1-\alpha)\,T^{-1}\,T_2\,\dot{T}\,. \tag{42}$$

The corresponding equations for elastic unloading can be obtained from [15]. The pertinent solution is

$$\dot{W}_e = (1-\beta^2\sin^2\theta)^{s/2}\cos[s(\omega-\pi)]\,, \tag{43}$$

where $\tan\omega = (1-\beta^2)^{\frac{1}{2}}\tan\theta$.

The problem is anti-symmetric with respect to $\theta = 0$, so only the domain $0 \le \theta \le \pi$ is considered. The boundary conditions are given by

$$\dot{W} = 0 \qquad \text{on } \theta = 0 \text{ (anti-symmetry)}\,,$$

$$\dot{W}'_e = 0 \qquad \text{on } \theta = \pi \text{ (zero tractions)} \tag{44}$$

As in the quasi-static solution [16], the possibility of a plastic reloading zone along the crack flanks is not considered. Although this may be a reasonable assumption for Mode III, results for the quasi-static elastic perfectly-plastic Mode I case [17] indicate that the above assumption is not correct in Mode I.

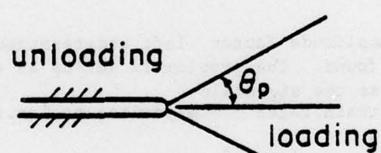

Fig. 3 Boundary between loading and unloading regions as $r \to 0$.

The boundary between loading and unloading zones surrounding the crack tip is assumed to be a radial line, emanating from the crack tip at an angle $\theta = \theta_p$, (Fig. 3). The field in the loading zone, $0 \le \theta \le \theta_p$, is governed by Eqns.(40) and (41-42) and the field in the elastic unloading zone $\theta_p \le \theta \le \pi$ is given by Eqn.(43).

To completely specify the problem, the continuity conditions at the interface, $\theta = \theta_p$, must be given. From the unloading condition, $\dot{\tau} = 0$, which implies $\dot{T} = 0$, we find

$$-\,s\,T\cos\theta + T'\sin\theta = 0 \qquad \text{at } \theta = \theta_p\,. \tag{45}$$

In addition, it is assumed that the particle velocity and stresses are continuous at $\theta = \theta_p$. This condition can be written as

$$[\dot{W}] = [\dot{W}'] = 0 \qquad \text{at } \theta = \theta_p\,, \tag{46}$$

where

$$[f(\theta)] = \lim_{\theta \to \theta_p^+} (f(\theta)) - \lim_{\theta \to \theta_p^-} (f(\theta)).$$

The first condition of Eqn.(46) can be automatically satisfied by writing the solution in the elastic unloading zone as

$$\dot{W}_e = \dot{W}(\theta_p^-)\, \dot{W}_e(\theta) \,/\, \dot{W}_e(\theta_p) \tag{47}$$

where $\dot{W}(\theta_p^-)$ is the solution in the loading region at $\theta = \theta_p$. From continuity of $\dot{W}'$ at $\theta = \theta_p$, it follows that

$$\dot{W}'(\theta_p^-) = \dot{W}(\theta_p^-)\, \dot{W}_e'(\theta_p) \,/\, \dot{W}_e(\theta_p), \tag{48}$$

where $\dot{W}_e(\theta_p)$ and $\dot{W}_e'(\theta_p)$ are found from Eqn.(43).

The problem has now been reduced to determining the solution in the plastic loading region, i.e. a solution satisfying the field equations in the region $0 \le \theta \le \theta_p$, and the boundary conditions at $\theta = 0$, θ_p. The quantities which have to be determined are $\dot{W}(\theta)$, $T_1(\theta)$, $T_2(\theta)$, θ_p and s. This is a nonlinear eigenvalue problem which must be solved numerically. The results for s and θ_p are shown in Tables 1 and 2, and further details of the analysis are given in [6].

The limit on v derived earlier, is an approximate agreement with the region where the numerical solution of the equations given above could be found.

TABLE 1. CALCULATED VALUES OF θ_p FOR DYNAMIC PLASTIC ANTI-PLANE SHEAR CRACK PROPAGATION

	θ_p				
α	β=0	β=0.1	β=0.25	β=0.5	β=0.75
1.0	1.571	1.576	1.602	1.690	1.786
0.7	1.522	1.528	1.554	1.643	1.731
0.5	1.473	1.478	1.505	1.595	
0.3	1.393	1.398	1.427	1.519	
0.2	1.328	1.334	1.363		
0.1	1.217	1.225	1.259		

TABLE 2. CALCULATED VALUES OF s FOR DYNAMIC PLASTIC ANTI-PLANE SHEAR CRACK PROPAGATION

	s				
α	β=0	β=0.1	β=0.25	β=0.5	β=0.75
1.0	-0.500	-0.500	-0.500	-0.500	-0.500
0.7	-0.444	-0.444	-0.442	-0.434	-0.396
0.5	-0.395	-0.394	-0.391	-0.375	
0.3	-0.325	-0.324	-0.319	-0.288	
0.2	-0.277	-0.276	-0.269		
0.1	-0.208	-0.206	-0.194		

MODE III STEADY-STATE DYNAMIC CRACK PROPAGATION IN AN ELASTIC PERFECTLY-PLASTIC MATERIAL

In the previous section, the case where the system of governing equations is elliptic, and a separation-of-variable type asymptotic analysis is applicable, was considered. In this section the special case of an elastic perfectly-plastic material, for which the governing equations in the plastic loading region are hyperbolic, is investigated. The analysis is again asymptotic in the sense that the solution is not valid far from the crack tip. However it is different from the approach where only terms of the highest singularity in r are retained.

The problem is a generalization to dynamics of the problem addressed by Chitaley and McClintock [9] for quasi-static crack growth. The Mode III case is used as a means to understand the importance of inertial terms on the crack tip fields, and to gain insight into the more difficult Mode I case. Although for an analysis of a real material, the exclusion of strain rate effects in the material model cannot be justified when other dynamic plasticity effects are included, this simplification is made to make the problem tractable to an asymptotic analysis. The results which are obtained on the basis of several assumptions do show that when compared to the quasi-static case [9], there is a major difference. As opposed to the cases discussed in the previous section it is not clear that the equations governing the asymptotic field for the transient case are the same as those for the steady-state case when the material behavior is elastic perfectly-plastic; that is, it cannot be assumed that $(\dot{\ }) \simeq -v\,\partial/\partial x_1$ is a good approximation for the transient case. However, only the steady-state case will be considered here.

Constitutive Law (Fig. 2d)

The yield condition is

$$\tau = \tau_o^2 . \tag{49}$$

Adopting a Prandtl-Reuss incremental flow law as in Eqn. (28), the strain rates are given by

$$\dot{\gamma}_i = \dot{\tau}_i/\mu + \dot{\lambda}\,\tau_i \ , \tag{50}$$

where $\dot{\lambda} > 0$ if $\tau = \tau_o$, $\dot{\tau} > 0$, otherwise $\dot{\lambda} = 0$. Equation (50) is a special case of Eqn.(28) but as the analysis is fundamentally different, the governing equations are rederived.

Governing Equations

Using the steady state form of Eqn.(23), i.e. all time derivatives, are assumed to vanish identically, the governing equations can be written in the following form,

$$\tau_{i,i} = \rho v^2 \partial^2 w/\partial x_1^2 \ , \qquad \text{(equilibrium)} \tag{51}$$

$$\gamma_{1,2} = \gamma_{2,1} \qquad \text{(compatability)} \tag{52}$$

$$\gamma_{i,1} = \tau_{i,1}/\mu + \lambda_{,1}\, \tau_i \qquad \text{(constitutive law)} \tag{53}$$

where $\lambda_{,1} < 0$ if $\tau = \tau_o$, $\tau_i\, \tau_{i,1} < 0$,

otherwise $\lambda_{,1} = 0$.

In the plastically deforming regions, where $\lambda_{,1} < 0$, the yield condition can be identically satisfied if we define ϕ such that

$$\tau_1 = -\tau_o \sin\phi \ , \ \tau_2 = \tau_o \cos\phi \ . \tag{54}$$

Substituting into Eqns.(51-53) and eliminating γ_2 , the governing equations can be written as

$$\begin{bmatrix} \beta^2 & \frac{\tau_o}{\mu}\cos\phi \\ \frac{\tau_o}{\mu}\cos\phi & \frac{\tau_o^2}{\mu^2} \end{bmatrix} \underline{w}_{,1} + \begin{bmatrix} 0 & \frac{\tau_o}{\mu}\sin\phi \\ \frac{\tau_o}{\mu}\sin\phi & 0 \end{bmatrix} \underline{w}_{,2} = \underline{0} \ ; \ \underline{w} = \begin{Bmatrix} \gamma_1 \\ \phi \end{Bmatrix} \tag{55}$$

where $\beta = v/\sqrt{\mu/\rho}$. Equation (55) has the general form $A^1\, \underline{w}_{,1} + A^2\, \underline{w}_{,2} = \underline{0}$.

Using the method of characteristics [13], Eqn.(55) is transformed into a set of ordinary differential equations along the characteristics in the $x_1 - x_2$ plane. To do this $\underline{w}_{,1}$ is written as $\underline{w}_{,1} = d\underline{w}/dx_1 - (dx_2/dx_1)\, \underline{w}_{,2}$ where $d\underline{w}/dx_1$ indicates differentiation along the characteristic whose tangent is given by dx_2/dx_1. Substituting this relation into Eqn. (55) results in

$$A^1\, d\underline{w}/dx_1 + (A^2 - A^1\, dx_2/dx_1)\, \underline{w}_{,2} = 0 \ . \tag{56}$$

Defining Λ as $\Lambda = dx_2/dx_1$ and solving for Λ from $|A^2 - \Lambda A^1| = 0$, which in this case is given by

$$\Lambda^2 \frac{\tau_o^2}{\mu^2}(\beta^2 - \cos^2\phi) + 2\Lambda \frac{\tau_o^2}{\mu^2}\sin^2\phi\cos^2\phi - \frac{\tau_o^2}{\mu^2}\sin^2\phi = 0 \ , \tag{57}$$

the characteristic directions are found to be

$$\left.\frac{dx_2}{dx_1}\right|_{\pm} = \frac{\sin\phi}{\cos\phi \pm \beta} \ , \tag{58}$$

Here the $\pm$ signs are used to denote the $\pm$ ve characteristic curves. The left eigenvectors, $\underline{\ell}$, corresponding to Λ, which are defined by $\underline{\ell}.[A^2 - \Lambda A^1] = 0$, are

$$\underline{\ell}_{\pm} = \left(\pm \frac{\tau_o}{\mu\beta}, 1\right) \tag{59}$$

If $\underline{\ell}$ is contracted with Eqn.(55), the result is an ordinary differential equation, called the characteristic relation, along each characteristic, viz,

$$\underline{\ell}.A^1 \, d\underline{w}/dx_1 + \underline{\ell}.(A^2 - \lambda A^1) \, \underline{w}_{,2} = \underline{\ell}.A^1 \, d\underline{w}/dx_1 = 0 \, . \tag{60}$$

For the problem at hand the characteristic relations are

$$\beta \, d\gamma_1 \pm (\tau_o/\mu) \, d\phi = 0 \tag{61}$$

Equation (61) can be integrated along the characteristics to give the Riemann invariants,

$$J_{\pm} = \beta\gamma_1 \pm (\tau_o/\mu) \, \phi \tag{62}$$

where $J_{\pm}$ are constants along the corresponding characteristics.

From Eqns.(61) and (62) a number of types of characteristic fields in a region in the $x_1 - x_2$ plane can be constructed. They are

a) simple field , J_+ = constant, J_- = constant,
both sets of characteristics straight;

b) uniform field, J_+ = constant, $J_- \neq$ constant,
-ve set of characteristics straight;

c) uniform field, $J_+ \neq$ constant, J_- = constant,
+ve set of characteristics straight;

and

d) non-uniform field, $J_+ \neq$ constant, $J_- \neq$ constant,
neither set of characteristics straight.

At this point, the difference between the governing equations in the plastic zones for the dynamic and quasi-static cases can be pointed out. If in Eqns.(61) and (62) the limit $\beta \to 0$ is taken, the two characteristics and the corresponding Riemann invariants degenerate to one characteristic and one Riemann invariant, and the appropriate quasi-static limit is obtained. The resulting set of equations implies that ϕ is constant along a characteristic and therefore the characteristics are straight. Unfortunately in the dynamic case under consideration, the characteristics are in general not straight since their direction at any point depends on the solution, ϕ , at that point.

Boundary Conditions

The Mode III problem is anti-symmetric about $x_2 = 0$, which implies $w = 0$ along $x_2 = 0$, $x_1 > 0$. Therefore $\gamma_1 = 0$ on the x_1 - axis ahead of the crack and by Eqn.(53), this implies $\tau_1 = 0$ or $\phi = 0$ on $x_2 = 0$, $x_1 > 0$.

On the crack flanks ($x_2 = 0$, $x_1 < 0$) we have $\tau_2 = 0$. The boundary conditions near the crack tip are then

$$\phi = 0 \quad , \qquad x_2 = 0 \, , \quad x_1 > 0 \, , \tag{63}$$

$$\phi = \pm \pi/2 \quad , \qquad x_2 = 0 \, , \quad x_1 < 0 \, . \tag{64}$$

In Eqn.(64), the choice of τ_1 being τ_o or $-\tau_o$ depends on the solution given below.

The boundary conditions far from the crack tip are not specified as they will not enter the analysis. However, it is envisioned that the small scale yielding assumption will hold and the plastic zones will be embedded in the linear elastic stress field for a moving crack.

Since the problem is anti-symmetric, only the upper-half plane, $x_2 > 0$, will be considered.

Near-Tip Fields

From Eqn.(62), a difference can be seen between the quasi-static and dynamic cases. From [9,12,17], the quasi-static strain γ_1 is found to be logarithmically singular as the crack tip is approached. However, in the dynamic case, if γ_1 should become unbounded as $r \to 0$ but ϕ is restricted to finite values, it follows from Eqn.(62) that $J_\pm$ must also tend to infinity. The only way this could happen with bounded ϕ is for ϕ to be cyclic in its permissible range. The latter would imply that the stresses must also be cyclic, which is unacceptable for a solution in a plastic zone Thus, γ_1 is assumed to be bounded in the dynamic case.

A solution is now constructed, which satisfies the boundary conditions on $x_2 = 0$, by using the simple and uniform fields a) and b) given above. Although it is possible that the near-tip plastic loading field may be non-uniform, this case is not considered. Case c) is ruled out since $\lambda_{-1} > 0$ and cases a) or b) taken separately cannot match both boundary conditions. The case of an elastic unloading zone is also considered, but will be shown not to exist to the order of approximation made.

After a process of elimination the following solution is found. The resulting characteristics are shown in Fig. 4.

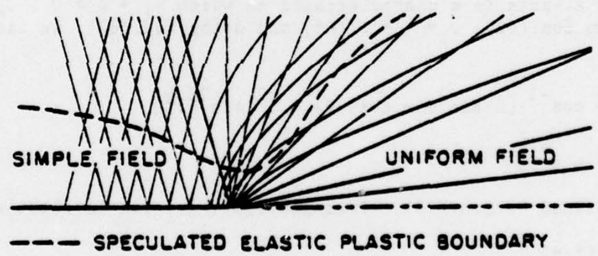

Fig. 4 Characteristic curves near the crack tip for Mode III steady-state dynamic crack propagation

Let a simple field border on $x_2 = 0$, $x_1 < 0$. Then, in this region the characteristics are straight, being given by

$$x_2 = \pm x_1/\beta + A_\pm \, , \tag{65}$$

where $A_\pm$ are constants and ϕ , γ_1 are constant. Applying the traction free crack face boundary condition given in Eqn.(64), results in $\phi = \pi/2$ in the simple field. The choice of $\phi > 0$ will be obvious once the field in front of the crack tip is found.

In front of the crack tip , on the x_1 - axis , $\phi = \gamma_1 = 0$ satisfies the characteristic relations in Eqn.(61). Therefore the x_1 - axis is a characteristic. Also, from the stress distribution in the linear elastic case, a plastic loading zone is expected to form ahead of the crack tip. Using this as motivation, case b), with the -ve characteristics straight and

forming a centred fan at the crack tip is used to match the boundary condition ahead of the crack tip. Case b) assumes J_+ is constant, which requires γ_1, ϕ be constant along a -ve characteristic. Substituting this into Eqn.(58), the -ve characteristics are found to be defined by

$$x_2 - x_1 \frac{\sin\phi}{\cos\phi - \beta} = x_2 - x_1 \tan\theta = 0 \tag{66}$$

where the constant of integration is zero to satisfy the centred fan condition, and θ is the angle between the characteristic and the positive x_1 - axis.

Differentiating Eqn.(66) with respect to x_1 and solving for ϕ in terms of θ from the resulting equation, we find

$$\cos\phi = \beta \sin^2\theta + \cos\theta\,(1 - \beta^2\sin^2\theta)^{\frac{1}{2}} . \tag{67}$$

Substituting this result into Eqns.(54) gives

$$\tau_1 = -\tau_o \sin\theta\,[(1 - \beta^2 \sin^2\theta)^{\frac{1}{2}} - \beta\cos\theta] \tag{68}$$

and

$$\tau_2 = \tau_o\,[\beta \sin^2\theta + \cos\theta\,(1 - \beta^2 \sin^2\theta)^{\frac{1}{2}}] . \tag{69}$$

In the simple field, $\tau_2 = 0$. The angle at which the uniform and simple fields match is found from setting Eqn.(69) to zero. The result is

$$\tan\theta^* = -1/\beta , \tag{70}$$

which defines a characteristic of both the simple and uniform fields emanating from the crack tip.

Since the x_1-axis is a characteristic on which $v_1 = \phi \equiv 0$, J_+ is zero. Therefore, from Eqn.(62), $\gamma_1 = -\tau_o\phi/(\mu\beta)$ and using Eqn.(67), we can write

$$\gamma_1 = -\frac{\tau_o}{\mu\beta} \cos^{-1}[\beta \sin^2\theta + \cos\theta\,(1 - \beta^2 \sin^2\theta)^{\frac{1}{2}}] , \tag{71}$$

for $0 \le \theta \le \theta^*$,

where θ^* is defined by Eqn.(70). From Eqn.(53), $\lambda_{,1}$ can be obtained as

$$\lambda_{,1} = \frac{\cos(\phi-\theta)}{\tau_o r} \frac{d\gamma_1}{d\theta} , \tag{72}$$

and by virtue of Eqn.(71) it can then be easily shown that $\lambda_{,1} < 0$ as required. By evaluating γ_1 at θ^*, its value in the simple field can be obtained as $\gamma_1 = (\pi\,\tau_o)/(2\mu\beta)$. If Eqn.(70) is substituted in Eqn.(68), it is found that $\tau_1 = -\tau_o$, $\phi = \pi/2$, as previously assumed.

The possibility of an elastic unloading zone extending to the crack tip between the two plastic fields must now be checked. If the material unloads such that $\lambda_{,1} > 0$, there will be a wake region with residual plastic strains behind the plastic loading zone. Obviously these residual plastic strains will only be functions of x_2. The governing equations in the wake region are

$$(1-\beta^2)\,\tau_{1,1} + \tau_{2,2} = 0 , \tag{73}$$

$$\tau_{1,2} - \tau_{2,1} = -\mu\,\gamma^p_{1,2} , \tag{74}$$

where γ_1^p is the non-recoverable part of the γ_1 strain. From Eqn.(71), evaluated at $\theta = \theta^*$, it can be seen that γ_1 and τ_1 at θ^* are not functions of x_2 and therefore in the elastic unloading region $\gamma_{1,2}^p$ is zero. No solution can then be found of Eqns.(73-74) which matches the continuity conditions on both sides of the elastic unloading region, and therefore the assumption of an unloading zone is incorrect.

From Eqn.(52), $\gamma_{2,1}$ can be found and after integrating with respect to x_1, γ_2 in the domain $0 \le \theta \le \theta^*$ is obtained as

$$\gamma_2 = \frac{\tau_o}{\mu}\left\{\frac{1-\beta}{2\beta} \ln\left[1 - \beta \sin^2\theta - \cos\theta\,(1 - \beta^2\sin^2\theta)^{\frac{1}{2}}\right]\right.$$

$$\left. + \frac{1+\beta}{2\beta} \ln\left[1 + \beta \sin^2\theta + \cos\theta\,(1-\beta^2\sin^2\theta)^{\frac{1}{2}}\right]\right\} + \varphi(x_2) \quad , \tag{75}$$

where $\varphi(x_2)$ will be specified later. By continuity of strains, for $\theta^* \le \theta \le \pi$

$$\gamma_2 = \varphi(x_2) \tag{76}$$

Requiring γ_2 to be bounded at $x_1 = r_p$ and for small x_2, where $x_1 = r_p$ is the intersection of the elastic-plastic boundary with the x_1- axis, $\varphi(x_2)$ can be approximately given by

$$\varphi(x_2) = \frac{\tau_o}{\mu}\left\{\frac{1-\beta}{\beta}\ \ell n\left(\frac{r_p}{x_2}\right) - \frac{1-\beta}{\beta}\ \ell n(1-\beta) - \ell n2 + 1\right\} . \tag{77}$$

From the γ_1 strain in the simple field, the crack opening angle is given by

$$\theta_{op} = 2 \tan^{-1}\left(\frac{\pi\ \tau_o}{2\mu\beta}\right) \tag{78}$$

and the crack opening displacement is zero. If r is the radial distance to a point on a +ve characteristic, the +ve characteristics are given by

$$\frac{dr}{r} = -\left(\frac{\beta \cos\theta + (1-\beta^2\sin^2\theta)}{2\beta\sin\theta}\right)^{\frac{1}{2}} d\theta \tag{79}$$

The above results are essentially the same as those found by Slepyan [18], who used an approach where only the most singular terms $(O(1/r))$ were retained in the governing equations. This approach can be shown to give the same result as when the field variables are considered to be functions of (x_1/x_2) only. Since the results given above agree with [18], they must also be solutions for the case where the only independent variable is (x_1/x_2).

It is intuitively reasonable that the steady-state dynamic results should reduce to the quasi-static results at small values of β. However the transition from dynamic to quasi-static does not seem to be uniform since if $\beta \to 0$, γ_1 and γ_2 increase beyond bounds.

In the quasi-static case [9,17], $\gamma_1 \simeq \ell n(r)\sin\theta$ as $r \to 0$ and there is a region of elastic unloading. As a material point passes the crack tip, the increments of γ_1 change sign and this can only be achieved by elastic unloading in the quasi-static case. The same effect is not found in the dynamic case, since the plastic loading region extends completely around the crack tip. An estimation of the relative zone size near the crack tip over which the dynamic results are valid can be found from Eqns.(55). Since inertial effects are proportional to β^2, the dynamic results will be important in a region of $O(\beta^2)$, indicated in Fig. 4 by the minimum distance between the speculated elastic-plastic boundary and the crack tip. Outside this region, unloading may occur and as $\beta \to 0$, the unloading region may approach the crack tip, resulting in a transition to the quasi-static result.

Table 3: Comparison of results from Ref.[12,9,19] and the present analysis

	stationary crack (static)	propagating crack (steady-state) quasi-static	dynamic
strain γ_2 $x_2 = 0,\ x_1 > 0$	$\gamma_o\left(\frac{r_p}{r}\right)$ Ref.[12]	$\gamma_o\left[1+\ln\left(\frac{r_p}{r}\right) + \frac{1}{2}\ln^2\left(\frac{r_p}{r}\right)\right]$ Ref.[12]	$\gamma_o\left\{1+\frac{1-\beta}{\beta}\ln\left(\frac{r_p}{r}\right)\right\}$
r_p	$\frac{1}{\pi}\left(\frac{K^S_{III}}{\tau_o}\right)^2$ Ref.[12]	$\sim \frac{1}{\pi}\left(\frac{K^S_{III}}{\tau_o}\right)^2$ Ref.[9]	$\sim \frac{1}{\pi}\left(\frac{K^D_{III}}{\tau_o}\right)^2 \frac{1}{(1-\beta^2)^{\frac{1}{2}}}$
crack-opening angle θ_{op}	$2\tan^{-1}\left(\frac{\tau_o}{\mu}\right)$ Ref.[12]	π Ref.[19]	$2\tan^{-1}\left(\frac{\pi\tau_o}{2\mu\beta}\right)$
crack-opening displacement	$\frac{2(K^S_{III})^2}{\pi\mu\tau_o}$ Ref.[12]	0 Ref.[19]	0

K^S_{III} = static or quasi-static stress intensity factor

K^D_{III} = dynamic stress intensity factor

$\gamma_o = \tau_o/\mu$

A comparison is made in Table 3 between the static, quasi-static and dynamic results for a semi-infinite crack and small scale yielding. The approximate plastic zone size, r_p, for the dynamic case is estimated by using the Irwin approach of an effective crack length [20] and does appear to be consistent with the quasi-static result, although this may be purely fortuitous. In the limit as $\beta \to 1$, $K^D_{III} \to 0$ so that $r_p \to 0$ which would be the correct limit for the supersonic case. The crack-opening angles and displacements are consistent with $\theta_{op} \to \pi$ as $\beta \to 0$. The γ_2 strains along the x-axis are not consistent, but this is not unexpected, as the region over which the dynamic solution dominates, tends to zero in the limit $\beta \to 0$. It is expected that the strain outside the region in which the present analysis is valid will tend to the quasi-static result.

The above analysis has taken no account of loading or elastic-plastic boundary effects and is based on a number of simplifying assumptions. By a process of elimination, the above solution has been found to be the only one satisfying these assumptions. Further detailed analysis requires the help of extensive numerical work to give guidance in the analytical work.

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